A NOTE ON DECOMPOSABLE TOPOLOGICAL SPACES AND SUB-TOPOLOGICAL SPACES

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Abstract: In this note we introduce the concept of a decomposable topological space and sub-topological spaces and investigate the fundamental concepts in classical topological spaces for the decomposable topological spaces and sub-topological spaces. We shall also investigate some basic results concerning decomposable topological spaces and sub-topological spaces.

Keywords: Decomposable topology; decomposable topological space; non-decomposable topology; non-decomposable topological space; sub-topology; sub-topological space.

1. INTRODUCTION

In recent years the concept of a single topological space has been extended to bi-topological space (a non-vacuous set X endowed with two topologies τ_1 *and* τ_2), tri-topological space (a non-vacuous set X endowed with three topologies τ_1, τ_2 *and* τ_3) and quad topological space (a non-vacuous set X endowed with four topologies τ_1, τ_2, τ_3 *and* τ_4). The concept of a bi-topological space was first introduced by Kelly [1], tri-topological space was initiated by Kovar [2] and quad-topological space was investigated by Mukundan [3]. In these new settings most of the concepts relating to general topology have been studied. On the other hand, partition of a non-vacuous set X is defined as the collection of non-vacuous subsets X_λ of X where λ belonging to the indexing set I such that,

$$X = \bigcup_{\lambda \epsilon I} X_\lambda$$
$$X_\lambda \neq X_\mu; \ \lambda \neq \mu, \lambda, \mu \epsilon I$$

Expressed somewhat differently, a partition of X is the result of splitting it, or subdividing it, into non-vacuous subsets in such a way that each element of X belongs to one and only one of the given subset [5]. Motivated by these notions, we may introduce a new notion called decomposable topology τ on a non-vacuous set X. Any topology τ on X is said to be *decomposable***,** if

$$\tau = \bigcup_{\lambda \epsilon I} \tau_\lambda$$

where each τ_λ is a topology on X and
$\tau_\lambda \neq \tau_\mu, \lambda \neq \mu, \lambda, \mu \epsilon I$

Each τ_λ is called a *sub-topology* of the decomposable topology τ. The pair (X, τ) is called *decomposable topological space* and each pair (X, τ_λ) is called *sub-topological space* of the decomposable topological space (X, τ). A topology which cannot be expressed in above sense is termed as *non-decomposable topology* and

X equipped with such topology is called *non-decomposable topological space.*

For if $X = \{a, b, c\}$ with topology $\tau = \{\emptyset, \{a\}, \{a, b\}, X\}$, then τ can be expressed as

$\tau = \{\phi, \{a\}, X\} \cup \{\phi, \{a, b\}, X\} \cup \{\phi, X\}$
$= \tau_1 \cup \tau_2 \cup \tau_3$, where
$\tau_1 = \{\emptyset, \{a\}, X\}$,
$\tau_2 = \{\emptyset, \{a, b\}, X\}$ and
$\tau_3 = \{\emptyset, X\}$ are topologies on X . Thus
$\tau = \bigcup_{\lambda=1}^{3} \tau_{\lambda.}$ Clearly $\tau_\lambda \neq \tau_\mu$ for $\lambda \neq \mu$. Hence τ is a decomposable topology and (X, τ) is a decomposable topological space. In this case there are three sub-topologies τ_1, τ_2 and τ_3 each of which is non-decomposable and hence three sub-topological spaces $(X, \tau_1), (X, \tau_2)$ and (X, τ_3).

Detail discussion about classical topology and classical topological spaces may be found in [4, 6, 7].

In this note we aim at to study some of the fundamental concepts pertaining to the classical topological spaces and investigate their validity in the decomposable topological spaces and sub-topological spaces. We shall also investigate some basic results concerning decomposable topological spaces and sub-topological spaces.

2. SOME BASIC RESULTS

Here for brevity sake we shall however adopt the term d-topology to mean decomposable topology and d-topological space to mean decomposable topological space. Likewise, by the term s-topology and s-topological space we shall mean sub-topology and sub-topological space. Symbolically d-topology and d-topological space will be represented by $\tau^d$and (X, τ^d) whereas for each $\lambda \epsilon I$ s-topology and s-topological space by τ_λ^s, (X, τ_λ^s) and in general s-topologies and s-topological spaces by $\tau_{\lambda \epsilon I}^s$ and $(X, \tau_{\lambda \epsilon I}^s)$.

Let
$X = \{a, b, c\}$ and
$\tau_1 = \{\phi, \{a\}, \{a, b\}, X\}$
$\tau_2 = \{\phi, \{b\}, \{a, b\}, X\}$
$\tau_3 = \{\phi, \{a\}, \{b\}, \{a, b\}, X\}$
be topologies on X.It can easily be verified that each topology is d-topology on X.Thus, there may be more than one d-topology on X and each d-topological space has several s-topological spaces. It is evident that each s-topology τ_λ of $\tau, \tau_\lambda \subseteq \tau$ where $\lambda \in I$ and each element of τ belongs to some s-topology of τ . The last fact is contrary to the concept of partition of a set. Clearly, the Sierpinski space, indiscrete topological space and a topological space whose topology has only three elements are non-d-topological spaces and as such do not possess s-topological spaces. However, the following example demonstrates that the above three topological spaces may be the s-topological spaces of a d-topological space:

Example 2.1 Let $X = \{1, 2\}$ with $\tau = \{\emptyset, \{1\}, [2], X\}$ be a topological space.
Since $\tau = \{\emptyset, \{1\}, X\} \cup \{\emptyset, \{2\}, X\} \cup \{\emptyset, X\}$, so it is a d-topology having Sierpinski topology, a topology containing three elements and indiscrete topology as s-topologies.

Remark 2.2 Each d-topological space is a topological space but the converse is not always true.

Theorem 2.3 If a d-topological space (X, τ^d) has n elements in its d-topology τ^d, then (X, τ^d) possesses n-1 s-topological spaces.

Proof. Let (X, τ^d) be a d-topological space having n elements in d-topology τ^d. Let
$\tau^d = \{\emptyset, X \{x_1\}, \{x_2\}, \{x_3\}, \dots, \{x_{n-2}\}\}$ (n elements).The s-topologies of τ^d are constructed as follows
$\tau_1^s = \{\emptyset, X\}$

ISSN 2229-5518

$\tau_2^s = \{\emptyset, \{x_1\}, X\}$
$\tau_3^s = \{\emptyset, \{x_2\}, X\}$
Continuing this process, we have
$\tau_{n-2}^s = \{\emptyset, \{x_{n-3}\}, X\}$
$\tau_{n-1}^s = \{\emptyset, \{x_{n-2}\}, X\}$
This shows that there are n-1 s-topologies of τ^d and hence n-1 s-topological spaces $(X, \tau_{\lambda\epsilon I}^s)$ where $I = \{1, 2, 3, \ldots, n-1\}$.

Remarks 2.4

(i) Each d-topology has at least four elements.

(ii) Each s-topology of a d-topology has at least two elements and at the most three elements.

(iii) Each s-topology is a non-decomposable topology and hence form a non-decomposable topological space.

Theorem 2.5 Let X be a non-vacuous set. If τ_1 and τ_2 are two non-d-topologies on X such that the union of distinct elements of τ_1 and τ_2 belong to $\tau_1 \cup \tau_2$, then (X, $\tau_1 \cup \tau_2$) is a d-topological space.

Proof. Let X be a non-vacuous set and let $\tau_1 = \{\phi, \{a\}, X\}$ and $\tau_2 = \{\phi, \{b\}, X\}$ where $a, b \,\epsilon\, X$ be two non-d-topologies on X. Since $\{a\} \neq \{b\}$ and $\{a\} \cup \{b\} = \{a, b\} \epsilon \tau_1 \cup \tau_2$, therefore $\tau_1 \cup \tau_2 = \{\phi, \{a\}, X\} \cup \{\phi, \{b\}, X\} \cup \{a, b\} = \{\phi, \{a\}, \{b\}, \{a, b\}, X\}$. Clearly $\tau_1 \cup \tau_2$ is a d-topology and consequently $(X, \tau_1 \cup \tau_2)$ is a d-topological space.

Remark 2.6 In general, for if τ_λ, τ_μ , $\lambda \neq \mu, \lambda, \mu \,\epsilon\, I$ are non-d-topologies on a non-vacuous set X such that the union of distinct elements of τ_λ and τ_μ belong to $\tau_\lambda \cup \tau_\mu$, then $(X, \tau_\lambda \cup \tau_\mu)$ is a d-topological space.

The generalization of above theorem is given in the form of following:

Theorem 2.7 Let $\tau_\lambda, \lambda\epsilon I$ be non-d-topologies on a non-vacuous set X such that $\cup_{\lambda\epsilon I} \tau_\lambda$ contain an arbitrary union of elements of $\cup_{\lambda\epsilon I} \tau_\lambda$, then $(X, \cup_{\lambda\epsilon I} \tau_\lambda)$ is a d-topological space.

Example 2.8 Consider $X = \{a, b, c, d, e\}$ with non-d-topologies
$\tau_1 = \{\phi, \{a\}, X\}, \tau_2 = \{\phi, \{c\}, X\}, \tau_3 = \{\phi, \{d\}, X\}, \tau_4 = \{\phi, \{e\}, X\}$.
Then for $I = \{1, 2, 3, 4\}$, $\cup_{\lambda\epsilon I} \tau_\lambda = \{\phi, \{a\}, \{c\}, \{d\}, \{e\}, X\} \cup \{\{a, c\}, \{a, d\}, \{a, e\}, \{c, d\}, \{c, e\}, \{d, e\}, \{a, c, d\}, \{a, c, e\}, \{a, d, e\}, \{c, d, e\}, \{a, c, d, e\}\}$ is a d-topology on X and hence $(X, \cup_{\lambda\epsilon I} \tau_\lambda)$ is a d-topological space.

Remark 2.9 With slight modification in the statement, Theorem 2.7 is also true for s-topologies of a d-topology on X.

3. FUNDAMENTAL CONCEPTS

Since d-topological spaces are the classical topological spaces exclusive of Sierpinski space, indiscrete topological space and those having three elements in its topology, therefore all concepts relating to the classical topological spaces hold equally good in d-topological spaces. Hence from now onward our focus will be on s-topological spaces. In particular we shall define and study the fundamental concepts of classical topological spaces for s-topological spaces of a d-topological space.

3.1 Open and Closed sets in s-topological space: Let (X, τ^d) be a d-topological space. Let $\tau_\lambda^s, \lambda\epsilon I$ be s-topologies of τ^d. Elements of each $\tau_\lambda^s, \lambda\epsilon I$ on X are called *s-open sets (or said to be* τ_λ^s*-open in X).* If $u \in \tau_\lambda^s$ for some $\lambda \in I$, then u is τ_λ^s -open. Since each $\tau_\lambda^s \subseteq \tau^d$, so $u\epsilon\tau_\lambda^s$ implies that $u\epsilon\tau^d$. Thus τ_λ^s -open sets are τ^d-open sets.

ISSN 2229-5518

It is clear that each open set u in d-topological space X belongs to some s-topology of d-topology τ^d, that is, if $u\epsilon\,\tau^d$ then $u\,\epsilon\tau_\lambda^s$ for some s-topology $\tau_\lambda^s,\ \lambda\epsilon I$.
It may be noticed that ϕ and X are the only clopen sets belonging to each s-topology of the given d-topology.

Definition 3.1.1 Let (X, τ^d) be a d-topological space with s-topologies $\tau_\lambda^s, \lambda\epsilon I$. A subset A of X such that $A\epsilon\tau_\lambda^s, \lambda\epsilon I$ is called *s-open set* (*or said to be* τ_λ^s*-open*) and its relative complement is called *s-closed set*.

Example 3.1.2 Let X = {a, b, c} with d-topology $\tau^d = \{\emptyset, \{a\}, \{a, b\}, X\}$, and s-topologies
$\tau_1^s = \{\emptyset, \{a\}, X\}$, $\tau_2^s = \{\emptyset, \{a, b\}, X\}$ and $\tau_3^s = \{\emptyset, X\}$.
Consider a subset A = {a, b}$\subseteq X$. We see that the set $A\epsilon\tau_2^s$, therefore A is $\tau_2^s -$ open. The s-closed set of A is the set{c}. However, if we let A= {c}, then
$A\notin \tau_\lambda^s, \lambda\epsilon I$ and in this case A is not s-open but s-closed.

The sets
$\emptyset, \{a\}, X$ are $\tau_1^s -$ open,
$\emptyset, \{a, b\}, X$ are $\tau_2^s -$ open
and $\emptyset, X$ are $\tau_3^s -$ open.
The sets
$\emptyset, \{b, c\}, X$ are $\tau_1^s -$ closed,
$\emptyset, \{c\}, X$ are $\tau_2^s -$ closed
and $\emptyset, X$ are τ_3^s closed.
We observe that there are subsets in s-topological spaces (X, $\tau_{\lambda\epsilon I}^s$) which are open as well as closed, neither open nor closed and open (respt. closed) but not closed (respt. open).

Definition 3.1.3 Let (X, τ^d) be a d-topological space with s-topologies $\tau_\lambda^s, \lambda\epsilon I$. Let A be a subset of X. The *s-closure* of A, denoted by *s-cl(A)* is defined as the intersection of all s-closed sets of X containing A. Thus, if $\{C_\alpha : \alpha\epsilon I\}$ is the collection of all s-closed sets in X containing A, then

$$s\text{-}cl(A) = \cap_{\alpha\epsilon I} C_\alpha$$

It is clear that $A \subseteq$ *s-cl(A)*. Also A is s-closed if and only if A = *s-cl(A)*.

A point $x\epsilon X$ is called an *s-closure point* (or an *s-adherent point*) of A if and only if $x\epsilon$ *s-cl(A)*. If *s-cl(A)* = X, then A is said to be *s-dense* in X.

In Example 3.1.2 let A = {a}$\subseteq X$. The set A is $\tau_1^s -$ open. The s-open sets of X in τ_1^s are $\emptyset$, {a}, and X and the s-closed sets are X, {b, c} and $\emptyset$. Then the only s-closed set containing A is X. Hence *s-cl(A)* = X. Consequently, A is s-dense in X. The points a, b and c are the s-closure points of A. In the same fashion one may check *s-cl(A)*, s-denseness and s-closure points of A in the remaining s-topological spaces.

3.2 Union and Intersection of s-topologies: In the classical case the intersection of two topologies is a topology. In the following theorem, we prove the same result for s-topologies of a d-topology.

Theorem 3.2.1 Let (X, τ^d) be a d-topological space. The intersection of any two s-topologies of τ^d is an s-topology.

Proof. Let τ_λ^s and τ_μ^s, $\lambda \neq \mu, \lambda, \mu \in I$ be s-topologies of τ^d. Since τ_λ^s and τ_μ^s are topologies, therefore $\tau_\lambda^s \cap \tau_\mu^s$is a s-topology for $\lambda \neq \mu$.

It is observed that the intersection of any two s-topologies and hence an arbitrary number of s-topologies of d-topology yield an indiscrete topology so we may state this observation in the form of following:

Proposition 3.2.2 If $\tau_\lambda^s, \lambda\epsilon I$ are s-topologies of a d-topology τ^don X , then

$\cap_{\lambda \epsilon I} \tau_\lambda^s = \{\phi, X\}$.

Theorem 3.2.1 implies that the intersection of any two s-topologies of a d-topology τ^d is an s-topology; however, the union of two s-topologies of a d-topology may not be an s-topology:

Example 3.2.3. For, if
$X = \{\alpha, \beta, \gamma\} and\ \tau_1^s = \{\phi, \{\alpha\}, X\}$,
$\tau_2^s = \{\phi, \{\beta\}, X\}$ be two of the s-topologies of the d-topology
$\tau^d = \{\phi, \{\alpha\}, \{\beta\}, \{\alpha, \beta\}, X\}$ on X .
Then $\tau_1^s \cup \tau_2^s = \{\phi, \{\alpha\}, \{\beta\}, X\}$. We notice that this union is neither a topology since $\{\alpha\} \cup \{\beta\} = \{\alpha, \beta\} \notin \tau_1^s \cup \tau_2^s$ nor an s-topology as it contains four elements.

Remark 3.2.4. Though the union of two s-topologies may not be an s-topology, however, the union of all s-topologies τ_λ^s, $\lambda \epsilon I$ of a d-topology τ^d is equal to τ^d, that is $\cup_{\lambda \varepsilon I} \tau_\lambda^s = \tau^d$.

The following theorem gives characterization of each s-topology of a d-topology τ^d defined on X.

Theorem 3.2.5. Let τ^d be a d-topology on X and let τ_λ^s be any s-topology of τ^d. Then

(i) any union of elements of τ_λ^s belongs to τ_λ^s
(ii) any intersection of elements of τ_λ^s belongs to τ_λ^s
(iii) $\emptyset$ and X belong to τ_λ^s

3.3 Comparable s-topologies: We say that two s-topologies τ_λ^s and τ_μ^s of a d-topology τ^d are *comparable* if either of the set inclusion $\tau_\mu^s \subseteq \tau_\lambda^s$ or $\tau_\lambda^s \subseteq \tau_\mu^s$ holds.
The s-topologies τ_λ^s and τ_μ^s are said to be *equal* (and we write $\tau_\lambda^s = \tau_\mu^s$) if and only if both contain precisely the same elements.

In our initial example of Section 2, s-topologies of τ_1 are
$\tau_1^{/} = \{\phi, \{a\}, X\}, \tau_1^{//} = \{\phi, \{a, b\}, X\}$,
$\tau_1^{///} = \{\phi, X\}$,
s-topologies of τ_2 are
$\tau_2^{/} = \{\phi, \{b\}, X\}, \tau_2^{//} = \{\phi, \{a, b\}, X\}$,
$\tau_2^{///} = \{\phi, X\}$,
s-topologies of τ_3 are
$\tau_3^{/} = \{\phi, \{a\}, X\}, \tau_3^{//} = \{\phi, \{b\}, X\}$,
$\tau_3^{///} = \{\phi, \{a, b\}, X\}, \tau_3^{////} = \{\phi, X\}$

We see that none of the s-topologies of different d-topologies are comparable except the indiscrete s-topology which is comparable with all other s-topologies.

3.4 Coarser and Finer s-topologies: Suppose that τ_λ^s and $\tau_\mu^s, \lambda, \mu \epsilon I$ are s-topologies of a d-topology τ^d on a given non-empty set X. If $\tau_\lambda^s \subseteq \tau_\mu^s$, then we say that τ_λ^s is c*oarser (weaker or smaller)* than τ_μ^s or alternatively τ_μ^s is *finer* (*stronger or larger*) than τ_λ^s .

Example 3.4.1 Consider the s-topologies of d-topologies as mentioned in sub-section 3.3.
We notice that an indiscrete s-topology of a d-topology on X is the coarsest topology since it is contained in every other s-topology. On the other hand unlike the classical case the discrete topology cannot be the finest topology since it is not an s-topology of the d-topology. To be more specific, no comparison can be carried out among all other s-topologies.

3.5 Concept of s-Neighborhood in s-topological spaces: Let (X, τ^d) be a d-topological space with sub-topological spaces $(X, \tau_{\lambda \epsilon I}^s)$. A set $A \subseteq X$ is called a $\tau_\lambda^s -$ *neighborhood* (or an *s-neighborhood*) of a point x in X if there exists an s- open set U such that $x \epsilon U \subseteq A$. We call the collection of all τ_λ^s-neighborhoods of x as τ_λ^s-

neighborhood (or *s-neighborhood*) *system* of x and is denoted by $\mathcal{N}^s_{(x)}$.

Example 3.5.1 Let X= {a, b, c, d} with τ^d = {∅, {a, b}, {c, d}, X} be a d-topological space having s-topologies τ^s_1 = {∅, {a,b}, X}, τ^s_2 = {∅, {c,d}, X}, τ^s_3 = {∅, X}.

Consider a set {a, b, c} in X. Since {a, b} is an s-open set and $a \in \{a,b\} \subseteq \{a,b,c\}$, hence {$a$, b, c} is a τ^s_1-neighborhood of the point a in X. Other τ^s_1-neighborhoods of the point a are {a, b}, {a, b, d} and X. Therefore $\mathcal{N}^s_{(a)}$= {{a, b}, {a, b, c}, {a, b, d}, X}} is the τ^s_1-neighborhood system of a.

We have the following:

Theorem 3.5.2 If (X, τ^d) is a d-topological space with s-topologies τ^s_λ , $\lambda \epsilon I$, then every s-open set is an s-neighborhood of some point in X.

The s-neighborhood in the above theorem is also termed as an *s-open neighborhood*.

3.6 Concept of s-Limit point of a set:

Let (X,τ^d) be a d-topological space with s-topological spaces (X, $\tau^s_{\lambda\epsilon I}$) and $A \subseteq X$. A point $x \in X$ is called an *s-limit* of A in (X, $\tau^s_{\lambda\epsilon I}$) if and only if every s-neighborhood N^s of x contain a point of A other than x. Symbolically, we write, $N^s \cap (A \backslash \{x\}) \neq \phi$

Note that it doesn't make a difference if we restrict the condition to s-open neighborhoods only. The equivalent term such as *s-accumulation point, s-cluster point* or *s-derived point* may also be used for s-limit point of a set. The set of all s-limit points of A is called an *s-derived set* of A and is denoted by $A^{/}_s$ or $A^{\sim}_s$ or A^d_s.

Example 3.6.1 Let X = {a, b, c} with $\tau^d = \{\phi, \{a], \{c\}. \{a,c\}, X\}$ be a d-topological space. Then $\tau^s_1 = \{\phi, \{a\}, X\}, \tau^s_2 = \{\phi, \{c\}, X\}$, $\tau^s_3 = \{\phi, \{a,c\}, X\}, \tau^s_4 = \{\phi, X\}$ are s-topologies of τ^d. Let $A = \{b,c\}$. In (X, τ^s_1) s-topological space,
s-neighborhoods of $a\epsilon X$ are {a}, {a, b}, {a, c}and X. Since $a\epsilon\{a\}$ but {a} does not contain a point of A, therefore a is not an s-limit point of A.
s-neighborhood of $b\epsilon X$ is X. Since $b\epsilon X$ and contains a point of A other than b, therefore b is an s-limit point of A.
s-neighborhood of $c\epsilon X$ is X. Since $c\epsilon X$ and contains a point of A other than c, therefore c is an s-limit point of A.Hence A has two s-limit points b and c with respect to s-topological space (X, τ^s_1). Similarly, in s-topological space (X, τ^s_2) s-limit points of A are a and b. In (X, τ^s_3), a and b are s-limit point of A. In (X, τ^s_4) a, b and c are s-limit points of A.

Remark 3.6.2 It may be noticed that the common s-limit point(s) of a set $A \subseteq X$ with respect to all s-topological spaces (X, $\tau^s_{\lambda\epsilon I}$) is/are the limit point(s) of the respective d-topological space (X, τ^d).

3.7 Concept of s-Interior point and s-Interior of a set: If (X, τ^d) is a d-topological space and $A \subseteq X$, then a point $a \in A$ is called an *s-interior point* of A if there exists an s-open set U such that $a \in U \subseteq A$. The set of all s-interior points of A is called the *s- interior* of A and is denoted by *s-Int(A)*.

Example 3.7.1 Consider the set X = {a, b, c} with d- topology

$\tau^d = \{\emptyset, \{a\}, \{a, b\}, X\}$.

The s-topologies of τ^d are

$\tau_1^s = \{\emptyset, \{a\}, X\}$, $\tau_2^s = \{\emptyset, \{a, b\}, X\}$,

$\tau_3^s = \{\emptyset, X\}$.

If we choose the set $A = \{a, c\} \subseteq X$, we note that $a \in A$ is an s-interior point of A if we let $U=\{a\} \in \tau_1^s$, since $a \in U=\{a\} \subseteq \{a, c\}=A$.

However, the point $c \in A$ is not an s-interior point with respect to the s-topologies. The only s-open set that contains c is $U=X$ and $c \in U = X \nsubseteq \{a, c\} = A$. Therefore *s-Int (A)* = $\{a\}$.

Remark 3.7.2 *s-Int(A)*$\subseteq A$. For, if $x \in$ *s-Int(A),* then x is the s-interior point of A i.e. there exists an s-open set U such that $x \in U \subseteq A$. It follows that $x \in A$.

The following theorem gives characterization of s-interior of a set in a d-topological space.

Theorem 3.7.3 Let $(X, \tau_{\lambda \epsilon I}^s)$ be s-topological spaces of a d-topological space (X, τ^d). Let $A \subseteq X$. Then

1. *s-Int(A)* = u where u is an s-open set contained in A
2. *s-Int(A)* is s-open
3. *s-Int(A)* is the largest open set contained in A
4. A is an s-open if and only if A = *s-Int(A)*

3.8 Concept of s-Exterior point and s-Exterior of a set: If (X, τ^d) is a d-topological space with s-topological spaces $(X, \tau_{\lambda \epsilon I}^s)$ and $A \subseteq X$, then a point $x \in X$ is called an *s-exterior point* of A if $X \setminus A$ is an s-neighborhood of x. The set of all s-exterior points of A is called the *s-exterior* of A and is denoted by *s-Ext(A).*

Example 3.8.1 For, if $X = \{1, 2, 3, 4, 5, 6\}$ with $\tau^d = \{\phi, \{1\}, \{1, 2, 3, 4\}, \{1, 5, 6\}, \{5, 6\}, \{2, 3, 4, 5, 6\}, X\}$ be a d-topological space having s-topologies

$\tau_1^s = \{\phi, \{1\}, X\}$,
$\tau_2^s = \{\phi, \{1, 2, 3, 4\}, X\}$,
$\tau_3^s = \{\phi, \{1, 5, 6\}, X\}$
$\tau_4^s = \{\phi, \{5, 6\}, X\}$,
$\tau_5^s = \{\phi, \{2, 3, 4, 5, 6\}, X\}$,
$\tau_6^s = \{\phi, X\}$.

Let $A = \{1, 2, 3\}$. Then $X \setminus A = \{4, 5, 6\}$. Since, we can not find an s-open set containing 4 such that it is contained in $X \setminus A$, so it is not an s-exterior of A. On the other hand 5, 6 are easily seen to be s-exterior points of A in s-topological space (X, τ_4^s). Hence *s-Ext(A)* = $\{5, 6\}$.

3.9 Concept of s-Boundary point of a set: If (X, τ^d) is a d-topological space with s-topological spaces $(X, \tau_{\lambda \epsilon I}^s)$ and $A \subseteq X$, then a point $x \in X$ is called an *s-boundary point* or an *s-frontier point* of A if and only if every s-open neighborhood of x contain at least one point of A and at least one point of $X \setminus A$. The set of all s-boundary points of A is called an *s-boundary* or an *s-frontier* of A and is denoted by *s-*δA or *s-Bd(A)* or *s-Fr(A).* Symbolically,

s-$\delta A = X \setminus$(*s-Int(A)* $\cup$ *s-Ext(A)).*

In example 3.8.1, we see that 1, 2, 3 and 4 are the s-boundary points of A in (X, τ_4^s) . Hence *s-* $\delta A = \{1, 2, 3, 4\}$.

Conclusion: In this work we introduced the concept of a decomposable topological space and its sub-topological spaces and observed that decomposable topological spaces are merely the classical topological spaces exclusive of Sierpinski's space, indiscrete topological space and topological spaces having three elements in its topology. Hence there is every reason to believe that most of the other concepts relating to classical topological spaces (separation axioms, compactness, connectedness etc.) are true for decomposable topological spaces. Some results concerning the formation of a decomposable topological space from non-decomposable topological spaces were also investigated. It was further observed that unlike decomposable topological spaces sub-topological spaces are constrained topological spaces so the fundamental concepts of classical topological spaces were required to be rephrased for sub-topological spaces. Due to such limitations some of the concepts of classical topological spaces may not be true in the setting of sub-topological spaces.

Acknowledgment: The author wish to thank Prof. Dr. G. A. Khan for his kind guidance and encouragement throughout the preparation of this manuscript, while reading towards getting BS degree in Mathematics at the University of Peshawar, Khyber Pakhtunkhwa, Pakistan.

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